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Dear Bruria:

This is to let you know that I have found a general formula for the value of an infinite recurrent determinant. If the generating function

$$f(\omega) = \sum_{-\infty}^{\infty} c_n e^{ni\omega} \tag{1}$$

of the determinant

$$\Delta_k = \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_{-1} & c_0 & \cdots & c_{k-2} \\ \vdots & & & \\ c_{1-k} & c_{2-k} & \cdots & c_0 \end{vmatrix} \tag{2}$$

is bounded, it may be expressed in the general form

$$f(\omega) = c\, e^{ni\omega + \eta_+(\omega) + \eta_-(\omega)} \tag{3}$$

where <u>c</u> is a constant, <u>n</u> an integer, and the functions

$$\begin{aligned} \eta_+(\omega) &= \sum_{1}^{\infty} a_n^{(+)} e^{ni\omega} \\ \eta_-(\omega) &= \sum_{1}^{\infty} a_n^{(-)} e^{-ni\omega} \end{aligned} \tag{4}$$

are analytic in their respective half-planes.

822/78/1-2-39

2.

If $c \neq 1$, the determinant diverges (either literally, or in the sense that no minor converges to any number other than zero). Henceforth we assume $c = 1$.

If $n \neq 0$, the determinant converges to zero. Accordingly, we concentrate on the case $n = 0$.

In the case where f is a quotient of two polynomials which have zeros in opposite half-planes:

$$
\begin{aligned}
e^{\eta_+} &= \prod_1^{n_1} \left(1 - \alpha_j e^{i\omega}\right) \\
e^{-\eta_-} &= \prod_1^{n_1} \left(1 - \beta_k e^{-i\omega}\right)
\end{aligned}
\tag{5}
$$

the identity

$$e^{-\eta_-} f = e^{\eta_+}$$

shows us how to combine columns so as to prove that the determinant converges in a finite number of steps:

$$\Delta_m = \Delta_{m+1} = \dots = \Delta_\infty$$

Moreover, Δ_m is obviously a polynomial function of each α, of order n or less.

$n \neq 0$ Now if $\alpha_1 = 1/\beta_j$, we can divide out a pair of factors in f, so in (5), and $\Delta_m = 0$. Hence

$$\Delta_m = F(\beta_1 \dots \beta_m) \prod_{j,k} \left(1 - \alpha_j \beta_k\right)$$

and the specialization to

$$\alpha_1 = \text{---} = \alpha_m = 0$$

gives us a triangular determinant

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{vmatrix}$$

whence $F \equiv 1$, and we have the final result

$$\Delta_\infty = \Delta_m = \prod_{j,k} \left(1 - \alpha_j \beta_k\right) \tag{5a}$$

3.

This result admits of considerable generalization. Quite particularly, if log f is analytic in a strip which contains the real axis, then exp (η_+) and exp $(-\eta_-)$ may be approximated by polynomials which have no wrong zeros, (the $\underline{m}$ first terms of the power series will do), in such a manner that the corresponding determinants converge to

In crystal statistics we meet with generating functions of the type

$$f(\omega) = \prod_j (1-\alpha_j e^{i\omega})^{n_j} \prod_k (1-\beta_k e^{-i\omega})^{m_k} \tag{6}$$

and the corresponding determinant converges to the value

$$\cancel{\Delta_\infty = \frac{1}{2\pi}\int_{\omega=0}^{2\pi} \eta_+ \, d\eta_-(\omega)} \quad \prod (1-\alpha_j\beta_k)^{-n_j m_k} \tag{6a}$$

The general formula is

$$\log \Delta_\infty = \frac{i}{2\pi}\int_{\omega=0}^{2\pi} \eta_+ \, d\eta_-(\omega) \tag{7}$$

By (6a) we get the degree of order from C. S. III without much trouble. It equals $(1-k^2)^{1/8}$ as before.

By the usual constructions of harmonic conjugates, the result (7) may be expressed directly in terms of $\underline{f}$, which then involves the double integral

$$\iint \left(\log f(\omega)/f(\omega')\right)^2 \frac{d\omega\, d\omega'}{\sin^2 \frac{1}{2}(\omega-\omega')}$$

I have even satisfied myself that we can get the first order minors of (2) without much trouble. Our previous construction of reciprocals will probably give us minors of any desired order. This gives us the average product of any set containing an $\underline{\text{odd}}$ number of spins in the ordered state.

4.

I have been content to find sufficient conditions for these results; God only knows when the mathematicians will find necessary ones. The concept of a determinant which diverges to zero is tentative but I believe it has merit.

There will be time for all these things. I want this to reach you as is before you do more computations the hard way.

With best regards,